Linear Analysis Background

The linear analysis calculations reduce the Bladed aeroelastic model to a linear system at each operating point requested by the user. The linear system of equations in state-space form is represented by

$$\dot{\underline{\mathbf{x}}} = \mathbf{A}\underline{\mathbf{x}} + \mathbf{B}\underline{\mathbf{u}} \quad (1)$$
$$\underline{\mathbf{y}} = \mathbf{C}\underline{\mathbf{x}} + \mathbf{D}\underline{\mathbf{u}} \quad (2)$$

with

$$\underline{\mathbf{x}} = \mathbf{x} - \mathbf{x_0}, \quad \underline{\mathbf{y}} = \mathbf{y} - \mathbf{y_0}, \quad \text{and} \quad \underline{\mathbf{u}} = \mathbf{u} - \mathbf{u_0}$$

where $\mathbf{x}$ is a vector of states representing the system, $\mathbf{u}$ is the vector of system inputs and $\mathbf{y}$ is the vector of system outputs. The normalised vectors $\underline{\mathbf{x}}$,$\underline{\mathbf{y}}$ and $\underline{\mathbf{u}}$ are representing the deviation from equilibrium.
The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ represent the linearised relationship between these vectors. This represents a simplification of the full Bladed model which uses a fully non-linear set of equations to calculate the state derivatives and outputs

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \mathbf{u}) \quad (3)$$
$$\mathbf{y} = h(t, \mathbf{x}, \mathbf{u}). \quad (4)$$

It is important to note that in order to enable proper linearised wind turbine dynamical systems, the following principles for preparing the model need to be considered:

- Azimuthal dependency shall be removed which includes wind shear, yaw, rotor imbalance, etc.
- Physical effects that cannot be linearised shall be removed, for instance wind turbulence, stick-slip, etc.

In Bladed, the states fall into two main categories:

1. **Elastodynamic**: These are the states that represent the structural modes of the system. Elastodynamic modes are governed by $\mathbf{2^{nd}}$ order equations of motion. Therefore, to be represented in state-space form, each mode is represented by two states – displacement and velocity. This also includes the principal rotor rigid body freedom.

2. **Aerodynamic**: These are primarily used to model dynamic stall and dynamic wake. These states are generally $\mathbf{1^{st}}$ order as they are concerned with time-lags.

With the aerodynamic model in versions 4.7 and earlier, aerodynamic states are not included in model linearization. In the aerodynamic formulation in version 4.8 and later, the user has the option to include the aerodynamic states or not.

To perform linear analysis, Bladed takes each operating point in turn and finds the steady-state conditions of the turbine (as per the initial conditions in time-domain runs). This means that the rotor is not accelerating and the modal deflections are such that the elastic loads balance the external loading. This defines the values for $\mathbf{x_0}$, $\mathbf{y_0}$ and $\mathbf{u_0}$, the principal equilibrium point about which everything is perturbed.

For each input or state, Bladed then makes a series of perturbations of increasing magnitude either side of the equilibrium point. The value of the state or input is artificially increased or decreased, the system is solved with these edited values and the state derivatives and outputs are recorded. The number of perturbations and the maximum perturbation magnitude can be defined by the user.

The elements of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ can then be derived by performing a linear regression of the state derivative against the input or state value at all its perturbed values and its equilibrium value. The gradient of the linear regression gives the value of the element. If the correlation coefficient is less than the `minimum correlation coefficient`, then the relationship is considered void, and a zero value is given to the element.

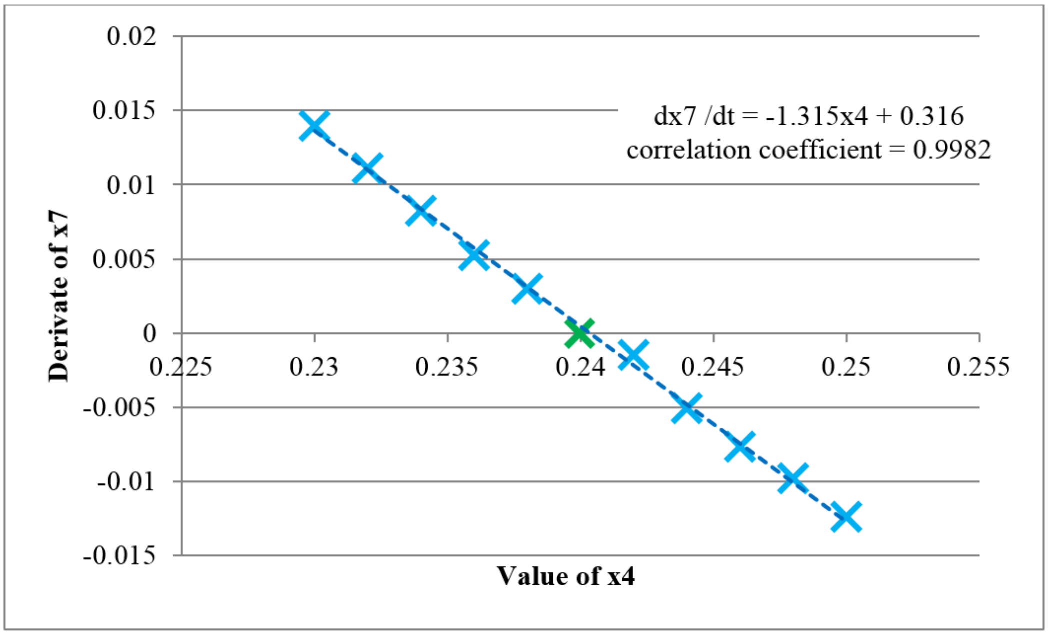

Figure 1: Example linear regression calculating element $\mathbf{A}_{7,4}$ with a value of -1.315, with a correlation coefficient of 0.9982. The equilibrium point is shown in green

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Multi-blade Coordinate Transformation

For linearisation calculations or Campbell diagrams it is recommended to select the multi-blade coordinate transformation, which generates coupled modes referring to the non-rotating coordinate system including the **backward and forward whirling** modes of the rotor. This is based on theory developed in (Bir, 2008) and (Hansen, 2003). The linearised model is significantly azimuth-dependent, but when transformed to non-rotating coordinates the resulting model of the structural dynamics should be only weakly azimuth-dependent. However, for 2-bladed turbines there is still a strong azimuth dependency.

Single mode transformation

The transformation matrix of displacements of a 3-blade system with azimuths ψ_1 to ψ_3 from non-rotating to rotating coordinates is given by

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \tilde{\mathbf{t}}_{NR\to R} \begin{bmatrix} q_0 \\ q_c \\ q_s \end{bmatrix}, \tag{1}$$

with

$$\tilde{\mathbf{t}}_{NR\to R} = \begin{bmatrix} 1 & \cos\psi_1 & \sin\psi_1 \\ 1 & \cos\psi_2 & \sin\psi_2 \\ 1 & \cos\psi_3 & \sin\psi_3 \end{bmatrix}. \tag{2}$$

Multi-blade coordinate transformations are often quoted in the above form, but the primary aim is to go the other way and transform from rotating to non-rotating coordinates. The transformation matrix of displacements of a 3-blade system from rotating to non-rotating coordinates is the inverse of the above matrix given by

$$\mathbf{t}_{R\to NR} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 2\cos\psi_1 & 2\cos\psi_2 & 2\cos\psi_3 \\ 2\sin\psi_1 & 2\sin\psi_2 & 2\sin\psi_3 \end{bmatrix}. \tag{3}$$

Note, that the inverse relation does not hold for the derivatives of this matrix.

The general transformation matrix for a turbine with an arbitrary number of blades (n) is

$$\mathbf{t}_{R\to NR} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2\cos\psi_1 & 2\cos\psi_2 & 2\cos\psi_3 & \cdots & 2\cos\psi_n \\ 2\sin\psi_1 & 2\sin\psi_2 & 2\sin\psi_3 & \cdots & 2\sin\psi_n \\ 2\cos j\psi_1 & 2\cos j\psi_2 & 2\cos j\psi_3 & \cdots & 2\cos j\psi_n \\ 2\sin j\psi_1 & 2\sin j\psi_2 & 2\sin j\psi_3 & \cdots & 2\sin j\psi_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & 1 & \cdots & (-1)^n \end{bmatrix}, \tag{4}$$

where the last row is the transformation to the differential mode and exists only if there is an even number of blades. For odd bladed turbines, the last row will be a sine cyclic row. The counter j

goes from 1 to $(n-1)/2$ if n is odd, and from 2 to $(n-2)/2$ if n is even.

Dropping the matrix representation the non-rotating coordinates can be calculated as

$$q_0 = \frac{1}{n}\sum_{i=1}^{n} q_i \tag{5}$$

$$q_{cj} = \frac{2}{n}\sum_{i=1}^{n} q_i \cos\left(j\psi_i\right) \tag{6}$$

$$q_{sj} = \frac{2}{n}\sum_{i=1}^{n} q_i \sin\left(j\psi_i\right) \tag{7}$$

$$q_d = \frac{1}{n}\sum_{i=1}^{n} q_i (-1)^n \tag{8}$$

Returning to the specific case of 3-bladed turbines as an example, the derivative transformation matrices are now calculated. Each azimuth angle ψ_i can be expressed in terms of the (assumed constant) rotor speed Ω and initial azimuth angle Ψ_i as linear relationship

$$\psi_i = \Omega t + \Psi_i. \tag{9}$$

Taking the time-derivatives of the transformation matrix gives

$$\dot{\mathbf{t}}_{R\to NR} = \frac{\Omega}{3}\begin{bmatrix} 0 & 0 & 0 \\ -2\sin\psi_1 & -2\sin\psi_2 & -2\sin\psi_3 \\ 2\cos\psi_1 & 2\cos\psi_2 & 2\cos\psi_3 \end{bmatrix}$$

and

$$\ddot{\mathbf{t}}_{R\to NR} = -\frac{\Omega^2}{3}\begin{bmatrix} 0 & 0 & 0 \\ 2\cos\psi_1 & 2\cos\psi_2 & 2\cos\psi_3 \\ 2\sin\psi_1 & 2\sin\psi_2 & 2\sin\psi_3 \end{bmatrix}$$

for the first and second derivatives, respectively.

System transformation matrix

A transformation matrix for the whole state list, including both displacement and velocity states, is required. For the displacement states we have already established in Equation (1) that

$$\mathbf{q}_{NR} = \mathbf{t}_{R\to NR}\mathbf{q}_R$$

holds. Taking the time-derivative of Equation (1) gives

$$\dot{\mathbf{q}}_{NR} = \mathbf{t}_{R\to NR}\dot{\mathbf{q}}_R + \dot{\mathbf{t}}_{R\to NR}\mathbf{q}_R \tag{10}$$

for the velocity states.

Combining $\mathbf{q}_{NR}$ and $\dot{\mathbf{q}}_{NR}$ to a vector of all states (both displacements and velocities) allows us to define a common transformation matrix $\mathbf{T}$ that is of the same dimensions as $\mathbf{A}$. We define

$$\mathbf{T} := \begin{bmatrix} \mathbf{t}_{R\to NR} & 0 \\ \dot{\mathbf{t}}_{R\to NR} & \mathbf{t}_{R\to NR} \end{bmatrix} \tag{11}$$

allowing us to express the combined vector as

$$\begin{bmatrix} \mathbf{q}_{NR} \\ \dot{\mathbf{q}}_{NR} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{R\to NR} & 0 \\ \dot{\mathbf{t}}_{R\to NR} & \mathbf{t}_{R\to NR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{R} \\ \dot{\mathbf{q}}_{R} \end{bmatrix}. \tag{12}$$

Note that in general the displacement and velocity states are not ordered in this way and a permutation of the system transformation matrix $\mathbf{T}$ will occur. The system transformation matrix is not singular and the inverse can be calculated.
The derivative of the system transformation matrix is trivially inferred as

$$\dot{\mathbf{T}} = \begin{bmatrix} \dot{\mathbf{t}}_{R\to NR} & 0 \\ \ddot{\mathbf{t}}_{R\to NR} & \dot{\mathbf{t}}_{R\to NR} \end{bmatrix}. \tag{13}$$

Transforming the A,B,C,D matrices

We consider the linear model equations in the rotating frame of reference and define

$$\mathbf{x}_R := \begin{bmatrix} \mathbf{q}_{R} \\ \dot{\mathbf{q}}_{R} \end{bmatrix} \tag{14}$$

to express the principal system as

$$\dot{\mathbf{x}}_R = \mathbf{A}_R\mathbf{x}_R + \mathbf{B}_R\mathbf{u} \tag{15}$$
$$\mathbf{y} = \mathbf{C}_R\mathbf{x}_R + \mathbf{D}_R\mathbf{u} \tag{16}$$

with respect to rotating blade coordinates.
The transformation of the state vector from rotating to non-rotating coordinates is given as

$$\mathbf{x}_{NR} = \mathbf{T}\mathbf{x}_R$$

and its derivative follows as

$$\dot{\mathbf{x}}_{NR} = \mathbf{T}\dot{\mathbf{x}}_R + \dot{\mathbf{T}}\mathbf{x}_R. \tag{17}$$

By combining Equation (17) with Equation (15) we infer

$$\dot{\mathbf{x}}_{NR} = \mathbf{T}\left(\mathbf{A}_R\mathbf{x}_R + \mathbf{B}_R\mathbf{u}\right) + \dot{\mathbf{T}}\mathbf{x}_R \tag{18}$$
$$= \left(\mathbf{T}\mathbf{A}_R + \dot{\mathbf{T}}\right)\mathbf{x}_R + \mathbf{T}\mathbf{B}_R\mathbf{u} \tag{19}$$
$$= \left(\mathbf{T}\mathbf{A}_R + \dot{\mathbf{T}}\right)\mathbf{T}^{-1}\mathbf{x}_{NR} + \mathbf{T}\mathbf{B}_R\mathbf{u} \tag{20}$$

and conclude

$$\mathbf{A}_{NR} = \left(\mathbf{T}\mathbf{A}_R + \dot{\mathbf{T}}\right)\mathbf{T}^{-1} \tag{21}$$
$$\mathbf{B}_{NR} = \mathbf{T}\mathbf{B}_R \tag{22}$$

from there.

Similar transformation in Equation (16) gives

$$\mathbf{y} = \mathbf{C}_R\mathbf{x}_R + \mathbf{D}_R\mathbf{u} \tag{23}$$

$$= \mathbf{C}_R\mathbf{T}^{-1}\mathbf{x}_{NR} + \mathbf{D}_R\mathbf{u} \tag{24}$$

for the output $\mathbf{y}$ of the linear system. We now define

$$\mathbf{C}_{NR} := \mathbf{C}_R\mathbf{T}^{-1}, \quad \text{and } \mathbf{D}_{NR} := \mathbf{D}_R \tag{25}$$

for the matrices concerned with the output of the linear model.
This completes the derivation of a linear model with respect to a non-rotating frame.

Rotational transformations are exclusively applied to states, which represent the degress of freedom in a mathematical model defined for all blades. These states include blade mode states as well as dynamic stall states, whereas any other individual-blade states such as pitch positions, rates, actuator internal states etc. are not transformed. The matrix $\mathbf{T}$ just has unit diagonal elements for rows and columns corresponding to the states and state derivatives which are not transformed. For other rows and columns, the elements of $\mathbf{T}$ represent the basic transformation defined above for each group of modes. Note that the elements connecting states and state derivatives also need to be defined by differentiating the equations of the basic transformation, bearing in mind that the derivative of the azimuth angle is equal to the rotor speed (which is assumed constant for this purpose). Model inputs and outputs are not transformed.

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Calculating Coupled Modes

The Campbell diagram and blade stability analyses are analyses of the matrix $\mathbf{A}$ at each specified operating point. Each coupled mode corresponds to an eigenvalue and its eigenvector. Given a (complex) eigenvalue, λ, of $\mathbf{A}$, Bladed reports the undamped frequency (ω_n), damped frequency (ω_d) and damping ratio (ζ) according to Figure 1.

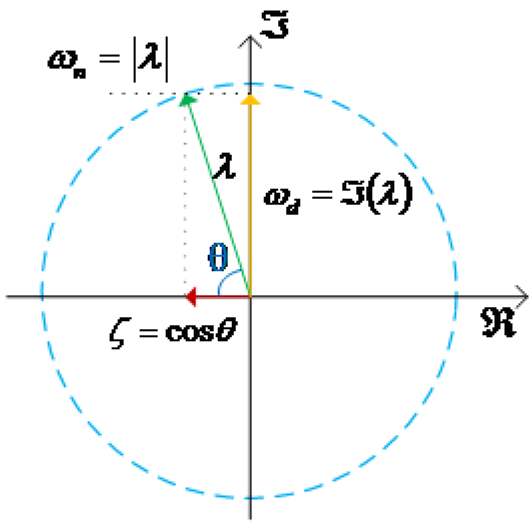

Figure 1: Argand Diagram

The uncoupled mode contributions to each coupled mode are determined by its eigenvector. If the coupled mode has contributions from second-order states (structural states), which are represented by two states in the state vector, then the displacement state is used to determine the contribution.

In their raw form, these eigenvector contributions represent the relative displacement of each mode and can be used to build up the coupled mode-shape. However, the contributions in the Campbell diagram have been normalised. This is done by modifying the matrix of eigenvectors such that each row and each column have a unit sum. This has the effect of increasing percentage contributions from modes with high mass and stiffness, which contribute very little in displacement but significantly in energy.

The phase of each contribution, ϕ_i, is determined by the argument of the corresponding complex eigenvector element, v_i, i.e.

$$\phi_i = \arctan\left(\frac{\mathrm{Im}\,(v_i)}{\mathrm{Re}\,(v_i)}\right).$$

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Naming Coupled Modes

In cases where coupled modes are computed such as in the Campbell diagram analysis the following sections gives details on the naming. A focus is placed on the behaviour when the multi-blade coordinate transform is used.

Support structure modes

For support structure modes, the coupled mode is named after the whole-tower mode that gives the highest contribution. Whole-tower modes are uniquely calculated for the linearisation calculations through a subsequent eigen analysis with fixed-free boundary conditions. This analysis considers the effect of the RNA and any other masses at distal nodes. In case multiple coupled support structure modes share the same whole-tower mode as its prime contributor, then the coupled mode name is made unique by appending letters A,B,C, and so on.

Rotor modes rotating frame

If no MBC transformation is used for the rotor modes, then the following logic applies to naming the coupled rotor modes:

- If a single blade mode gives >75% contribution to the coupled rotor mode, then the coupled rotor mode is named after that blade mode. In other words, the mode is called “Blade” instead of “Rotor” mode.
- Else, the rotor mode is named after its prime contributor and made unique by appending letters A,B,C, etc. in case multiple coupled rotor modes share the same uncoupled blade mode as prime contributor

Rotor modes non-rotating frame

If an MBC transform is applied then the individual blade modes are transformed to a set of rotor modes. For a three bladed rotor there typically is a collective, cosine-cyclic and sine-cyclic rotor mode. The 1st flapwise modes of all blades will be renamed to rotor 1st flapwise collective, rotor 1st flapwise sine-cyclic and rotor 1st flapwise cosine cyclic. In case the number of blades is even there will be a differential mode as well.

After the transformation and renaming of the individual blade modes the coupled rotor modes are named. The whirling modes are identified following the logic in the table below.

Coupled mode name	1st uncoupled mode	2nd uncoupled mode	Phase angle (ϕ_2 - ϕ_1)
Forward whirl	Sine cyclic	Cosine cyclic	> 0.0
	Cosine cyclic	Sine cyclic	< 0.0
Backward whirl	Sine cyclic	Cosine cyclic	< 0.0
	Cosine cyclic	Sine cyclic	> 0.0

If a coupled mode does not meet the criteria of the whirling modes, then the mode is named after its prime contributor. This is analogous with the naming logic of rotor modes in the rotating frame and support structure modes

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Joining Coupled Modes across Operating Points

The Campbell diagram displays the frequencies of different coupled displacement modes with respect to the rotor speed together with the most important excitation frequencies given in terms of multiples of the rotor frequency (P). In addition to the frequencies of the coupled modes, the Campbell diagram displays the corresponding damping ratios, which include the effect of structural damping as well as aerodynamic damping. Both characteristics are calculated as described in the article Calculating Coupled Modes and are useful for identifying the critical operating points that need further analysis.

The Joining Process

Given a set of coupled modes at each operating point, a fundamental step in creating the resulting Campbell diagram is to identify similar modes at adjacent operating points, which allows for joining similar modes across the operating points with line segments, giving the user the impression of continuous change in frequency against rotor speed (or wind speed). This joining process is generally challenging because the coupled modes evolve and change in their contributions between the operating points.

Similar coupled modes at two adjacent operating points are identified by comparing their complex eigenvectors and frequency in term of the extended modal assurance criterion (MACX) (Vacher, Jacquier, and Buchales, 2010) with frequency weighting. More specifically, the frequency weighted MACX numbers are calculated for all combinations of coupled modes at the two operating points to form a score matrix, which are then used for joining the modes by the Gale-Shapley algorithm (Gale & Shapley, 1962). A sequence of similar modes at all operating points forms a coupled mode series, which represents a line in the resulting Campbell diagram.

To ensure that the coupled mode series in the resulting Campbell diagram primarily involves structural dynamics, an initial calculation of coupled modes with only structural states is performed at the first operating point. These structure-only modes are then joined with the coupled modes at the first operating point as described above, which effectively excludes coupled modes that mainly have contributions from aerodynamic states.

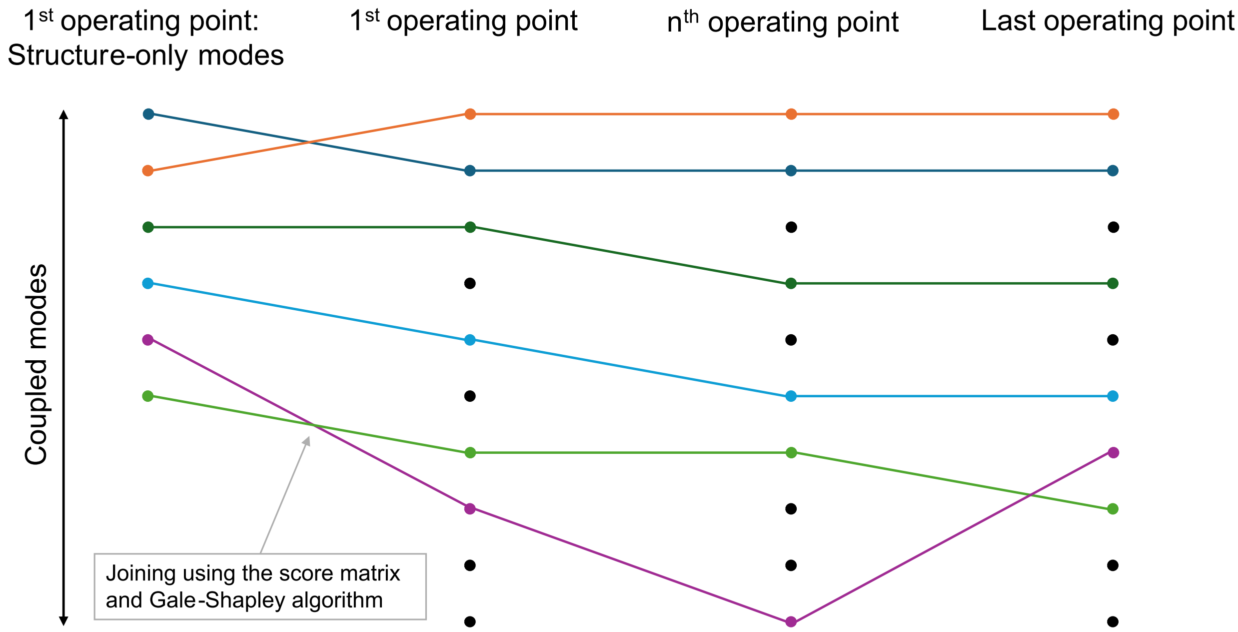

Figure 1: Illustration of the joining process showing how the coupled modes are connected at the operating points. Dots represent coupled modes. Coloured lines represent coupled mode series.

The resulting joining process is then:

1. Calculate a set of structure-only modes at the first operating point. These modes will also form the basis of the mode series, representing the lines in the resulting Campbell diagram.
2. Join the structure-only modes with the coupled modes at the first operating point. It is noted that the number of coupled modes is generally larger than the number of mode series, which means that not all coupled modes are included in a mode series.
3. Join the coupled modes that were included in a mode series at the current point (first operating point) with the coupled modes at the next point (second operating point). Repeat until the last operating point is reached.

Naming and Exclusion of Coupled Mode Series

A coupled mode series is named according to the contributions of the structure-only coupled modes at the first operating point only (more details on coupled mode naming can be found here). The contributions and therefore the shape of a coupled mode can change significantly between the range of operating points, and therefore the characteristic of a mode cannot be determined from the name alone.

A coupled mode series, which includes coupled modes with real eigenvalues (and therefore no oscillatory behaviour) at all operating points, is excluded from the resulting Campbell diagram. This is done because such modes cannot cause resonant behaviour, which is the primary purpose of the Campbell diagram to detect.

A coupled mode series is also excluded if the coupled mode frequencies at all operating points exceed a user-defined maximum value.

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Align Wind Field with Hub Axis

Figure 1 illustrates that azimuthal dependency is still invoked in certain cases even though the considerations given in the linear analysis background are followed. This occurs when a tilt angle is considered in the calculations and when the tower flexibility is strongly affecting the rotor orientation. Figure 1 (left) provides an illustration of the azimuthally independent system, indicating that the solution will be the same regardless of the azimuth angle of the rotor. However, when the tilt angle is considered, Figure 1 (middle), the loads perceived by the upper side of the rotor and the lower side of the rotor vary. This creates an imbalance of the loads similar as adding "a virtual wind shear" on the rotor plane. The situation is worse when the flexibility of the tower affects the rotor orientation strongly as illustrated in Figure 1 (right). Here, one can see that the hub orientation might be tilted even greater which creates a stronger loads imbalance across the rotor.

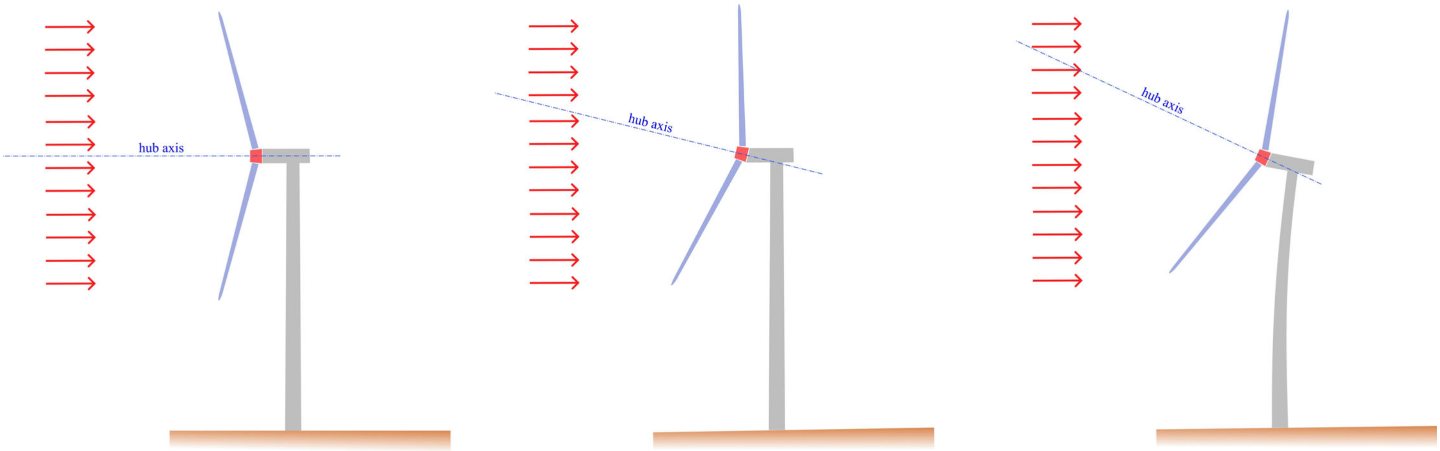

Figure 1: Illustration of the hub axis orientation relative to the incoming wind direction in the linearisation calculation. Left: rigid wind turbine without tilt angle, middle: rigid wind turbine with tilt angle being considered, right: flexible turbine with tilt angle being considered.

To avoid the imbalance of the loads due to the tilted hub orientation, Bladed introduces a correction to align the wind field according to the tilted hub axis orientation. The mechanism is clearly illustrated in Figure 2. It can be observed that when the wind field is aligned with the tilted hub axis, the loads experienced by the rotor will be independent of the azimuth angle. This effectively removes the azimuthal dependency of the system due to tilted hub axis orientation. This correction may also be applied for floating wind turbines as illustrated in Figure 3.

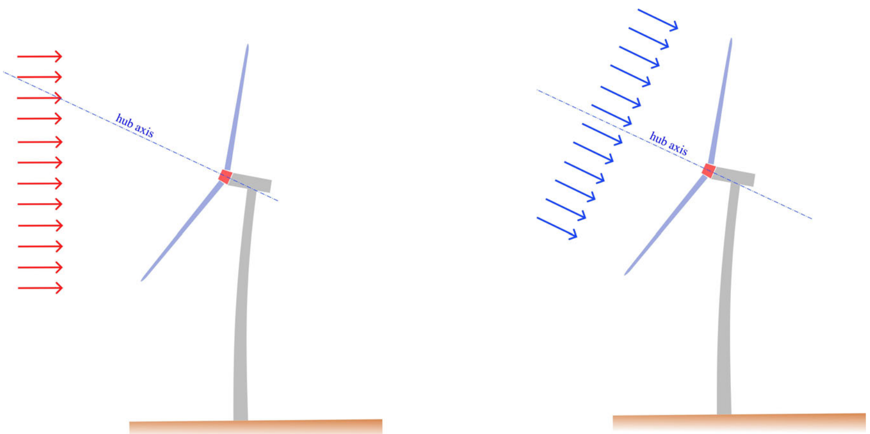

Figure 2: Aligning wind field to the tilted hub axis orientation for an onshore wind turbine or a bottom-fixed offshore wind turbine. Left: Azimuthally dependent system, right: azimuthally independent system.

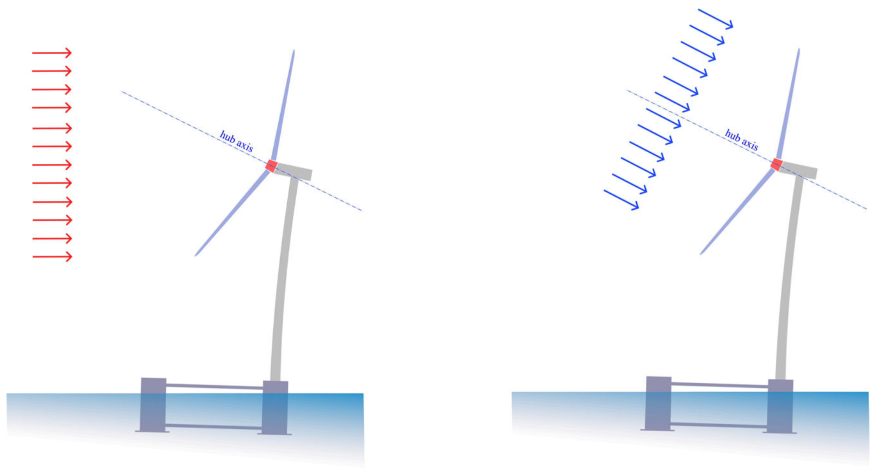

Figure 3: Aligning wind field to the tilted hub axis orientation for a floating wind turbine. Left: Azimuthally dependent system, right: azimuthally independent system.

The hub axis is determined during the initial conditions routine, where the rotor is not accelerating and the modal deflections are such that the elastic loads balance the external loading. Within the initial conditions routine, iterations are performed and the wind field is adjusted accordingly for every iteration to be parallel to the tilted hub axis. After the the conditions are found, the wind field orientation is fixed to the last found tilted angle in the initial conditions routine. Then, the system is perturbed using the fixed wind field orientation for calculating any of the linearisation type calculations available in Bladed. This process is done for every operating point simulated.

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