

The following are transformation equations defining the angular orientation of each coordinate system inherent in FAST.

Before providing these, it is useful to discuss the transformation equation relating coordinate system $\mathbf{x}$ to coordinate system $\mathbf{X}$ where $\mathbf{x}$ (with orthogonal axes $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$) is the coordinate system resulting from three rotations (θ_1, θ_2 , and θ_3) about the orthogonal axes ($\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$) of coordinate system $\mathbf{X}$. With all rotation angles assumed to be small, the order of rotations does not matter and Euler angles do not need to be used. Instead, what we want, is a transformation equation that is consistent with classical Bernoulli-Euler beam theory (which assumes small rotations). The correct transformation equation is:

$$\begin{Bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{Bmatrix} \approx \underbrace{\begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}}_{[A]} \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{Bmatrix},$$

where $[A]$ is referred to as the Bernoulli-Euler transformation matrix in this work. The approximation symbol ($\approx$) is used in place of an equals symbol ($=$) in the above expression since $[A]$ is not orthonormal, which implies that the resulting $\mathbf{x}$ from this expression is not made up of a set of mutually orthogonal axes (all transformation matrices between sets of mutually orthogonal axes must be orthonormal). So it is evident that in place of $[A]$, what we want is the closest orthonormal matrix to $[A]$, which is referred to as $[TransMat]$ in this work. From linear algebra, we know that the closest orthonormal matrix to $[A]$ in the Frobenius Norm sense is:

$$[TransMat] = [U][V]^T,$$

where the columns of $[U]$ contain the eigenvectors of $[A][A]^T$ and the columns of $[V]$ contain the eigenvectors of $[A]^T[A]$. This result follows directly from the Singular Value Decomposition (SVD) of $[A]$:

$$[A] = [U][\Sigma][V]^T,$$

where $[\Sigma]$ is a diagonal matrix containing the singular values of $[A]$, which are $\sqrt{\text{eigenvalues of } [A][A]^T} = \sqrt{\text{eigenvalues of } [A]^T[A]}$.

The algebraic form of the resulting transformation matrix is:

$$\begin{Bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\theta_1^2 \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} + \theta_2^2 + \theta_3^2}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} & \frac{\theta_3(\theta_1^2 + \theta_2^2 + \theta_3^2) + \theta_1 \theta_2 (\sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} - 1)}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} & \frac{-\theta_2(\theta_1^2 + \theta_2^2 + \theta_3^2) + \theta_1 \theta_3 (\sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} - 1)}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} \\ \frac{-\theta_3(\theta_1^2 + \theta_2^2 + \theta_3^2) + \theta_1 \theta_2 (\sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} - 1)}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} & \frac{\theta_1^2 + \theta_2^2 \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} + \theta_3^2}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} & \frac{\theta_1(\theta_1^2 + \theta_2^2 + \theta_3^2) + \theta_2 \theta_3 (\sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} - 1)}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} \\ \frac{\theta_2(\theta_1^2 + \theta_2^2 + \theta_3^2) + \theta_1 \theta_3 (\sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} - 1)}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} & \frac{-\theta_1(\theta_1^2 + \theta_2^2 + \theta_3^2) + \theta_2 \theta_3 (\sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2} - 1)}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} & \frac{\theta_1^2 + \theta_2^2 + \theta_3^2 \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}}{(\theta_1^2 + \theta_2^2 + \theta_3^2) \sqrt{1+\theta_1^2+\theta_2^2+\theta_3^2}} \end{bmatrix}}_{[TransMat]} \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{Bmatrix}$$

This was derived symbolically by J. Jonkman by computing $[U][V]^T$ by hand with verification in Mathematica.

Tower Base / Platform Coordinate System

$$\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix} = [TransMat(\theta_1 = q_R, \theta_2 = q_Y, \theta_3 = -q_P)] \begin{Bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{Bmatrix}$$

Tower Element-Fixed Coordinate System

$$\begin{Bmatrix} \mathbf{t}_1(h) \\ \mathbf{t}_2(h) \\ \mathbf{t}_3(h) \end{Bmatrix} = [TransMat(\theta_1 = \theta_{SS}(h), \theta_2 = 0, \theta_3 = \theta_{FA}(h))] \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix}$$

where,

$$\theta_{FA}(h) = -\left[\frac{d\phi_1^{TFA}(h)}{dh} q_{TFA1} + \frac{d\phi_2^{TFA}(h)}{dh} q_{TFA2} \right] \text{ and } \theta_{SS}(h) = \left[\frac{d\phi_1^{TSS}(h)}{dh} q_{TSS1} + \frac{d\phi_2^{TSS}(h)}{dh} q_{TSS2} \right]$$

Tower-Top / Base Plate Coordinate System

$$\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \left[TransMat(\theta_1 = \theta_{SS}(TwrFlexL), \theta_2 = 0, \theta_3 = \theta_{FA}(TwrFlexL)) \right] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

where,

$$\theta_{FA}(TwrFlexL) = - \left[\frac{d\phi_1^{TFA}(h)}{dh} \right]_{h=TwrFlexL} q_{TFA1} + \left[\frac{d\phi_2^{TFA}(h)}{dh} \right]_{h=TwrFlexL} q_{TFA2} \quad \text{and} \quad \theta_{SS}(TwrFlexL) = \left[\frac{d\phi_1^{TSS}(h)}{dh} \right]_{h=TwrFlexL} q_{TSS1} + \left[\frac{d\phi_2^{TSS}(h)}{dh} \right]_{h=TwrFlexL} q_{TSS2}$$

Nacelle / Yaw Coordinate System

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{bmatrix} \cos(q_{Yaw}) & 0 & -\sin(q_{Yaw}) \\ 0 & 1 & 0 \\ \sin(q_{Yaw}) & 0 & \cos(q_{Yaw}) \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Rotor-Furl Coordinate System

$$\begin{Bmatrix} rf_1 \\ rf_2 \\ rf_3 \end{Bmatrix} = \begin{bmatrix} [1 - \cos^2(RFrSkew)\cos^2(RFrTilt)]\cos(q_{RFrl}) & \cos(RFrSkew)\cos(RFrTilt)\sin(RFrTilt)[1 - \cos(q_{RFrl})] & \cos(RFrSkew)\sin(RFrSkew)\cos^2(RFrTilt)[\cos(q_{RFrl}) - 1] \\ +\cos^2(RFrSkew)\cos^2(RFrTilt) & -\sin(RFrSkew)\cos(RFrTilt)\sin(q_{RFrl}) & -\sin(RFrTilt)\sin(q_{RFrl}) \\ \cos(RFrSkew)\cos(RFrTilt)\sin(RFrTilt)[1 - \cos(q_{RFrl})] & \cos^2(RFrTilt)\cos(q_{RFrl}) + \sin^2(RFrTilt) & \sin(RFrSkew)\cos(RFrTilt)\sin(RFrTilt)[\cos(q_{RFrl}) - 1] \\ +\sin(RFrSkew)\cos(RFrTilt)\sin(q_{RFrl}) & & +\cos(RFrSkew)\cos(RFrTilt)\sin(q_{RFrl}) \\ \cos(RFrSkew)\sin(RFrSkew)\cos^2(RFrTilt)[\cos(q_{RFrl}) - 1] & \sin(RFrSkew)\cos(RFrTilt)\sin(RFrTilt)[\cos(q_{RFrl}) - 1] & [1 - \sin^2(RFrSkew)\cos^2(RFrTilt)]\cos(q_{RFrl}) \\ +\sin(RFrTilt)\sin(q_{RFrl}) & -\cos(RFrSkew)\cos(RFrTilt)\sin(q_{RFrl}) & +\sin^2(RFrSkew)\cos^2(RFrTilt) \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

Shaft Coordinate System

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} \cos(ShftSkew)\cos(ShftTilt) & \sin(ShftTilt) & -\sin(ShftSkew)\cos(ShftTilt) \\ -\cos(ShftSkew)\sin(ShftTilt) & \cos(ShftTilt) & \sin(ShftSkew)\sin(ShftTilt) \\ \sin(ShftSkew) & 0 & \cos(ShftSkew) \end{bmatrix} \begin{Bmatrix} rf_1 \\ rf_2 \\ rf_3 \end{Bmatrix}$$

Azimuth Coordinate System

$$\begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(q_{DrTr} + q_{GeAz}) & \sin(q_{DrTr} + q_{GeAz}) \\ 0 & -\sin(q_{DrTr} + q_{GeAz}) & \cos(q_{DrTr} + q_{GeAz}) \end{bmatrix} \begin{Bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{Bmatrix}$$

Teeter Coordinate System

$$\begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{Bmatrix} = \begin{bmatrix} \cos(q_{Teet}) & 0 & -\sin(q_{Teet}) \\ 0 & 1 & 0 \\ \sin(q_{Teet}) & 0 & \cos(q_{Teet}) \end{bmatrix} \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix}$$

Hub / Delta-3 Coordinate System

$$\begin{Bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Delta 3) & \sin(\Delta 3) \\ 0 & -\sin(\Delta 3) & \cos(\Delta 3) \end{bmatrix} \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{Bmatrix}$$

Hub (Prime) Coordinate System

$$\begin{Bmatrix} \mathbf{g}_1^{B1} \\ \mathbf{g}_2^{B1} \\ \mathbf{g}_3^{B1} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{Bmatrix}$$

The equation for $\mathbf{g}^{B2}$ of blade 2 is similar.

Coned Coordinate System

$$\begin{Bmatrix} \mathbf{i}_1^{B1} \\ \mathbf{i}_2^{B1} \\ \mathbf{i}_3^{B1} \end{Bmatrix} = \begin{bmatrix} \cos[\text{PreCone}(l)] & 0 & -\sin[\text{PreCone}(l)] \\ 0 & 1 & 0 \\ \sin[\text{PreCone}(l)] & 0 & \cos[\text{PreCone}(l)] \end{bmatrix} \begin{Bmatrix} \mathbf{g}'_1^{B1} \\ \mathbf{g}'_2^{B1} \\ \mathbf{g}'_3^{B1} \end{Bmatrix}$$

The equation for $\mathbf{i}^{B2}$ is similar.

Blade / Pitched Coordinate System

$$\begin{Bmatrix} \mathbf{j}_1^{B1} \\ \mathbf{j}_2^{B1} \\ \mathbf{j}_3^{B1} \end{Bmatrix} = \begin{bmatrix} \cos[\text{BlPitch}(l)] & -\sin[\text{BlPitch}(l)] & 0 \\ \sin[\text{BlPitch}(l)] & \cos[\text{BlPitch}(l)] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{i}_1^{B1} \\ \mathbf{i}_2^{B1} \\ \mathbf{i}_3^{B1} \end{Bmatrix}$$

The equation for $\mathbf{j}^{B2}$ is similar.

Blade Coordinate System Aligned with Local Structural Axes (not element fixed)

$$\begin{Bmatrix} \mathbf{Lj}_1^{B1}(r) \\ \mathbf{Lj}_2^{B1}(r) \\ \mathbf{Lj}_3^{B1}(r) \end{Bmatrix} = \begin{bmatrix} \cos[\theta_s^{B1}(r)] & -\sin[\theta_s^{B1}(r)] & 0 \\ \sin[\theta_s^{B1}(r)] & \cos[\theta_s^{B1}(r)] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{i}_1^{B1} \\ \mathbf{i}_2^{B1} \\ \mathbf{i}_3^{B1} \end{Bmatrix}$$

The equation for $\mathbf{Lj}^{B2}(r)$ is similar.

Blade Element-Fixed Coordinate System Aligned with Local Structural Axes

$$\begin{Bmatrix} \mathbf{n}_1^{B1}(r) \\ \mathbf{n}_2^{B1}(r) \\ \mathbf{n}_3^{B1}(r) \end{Bmatrix} = \left[\text{TransMat}(\theta_l = \theta_x^{B1}(r), \theta_2 = \theta_y^{B1}(r), \theta_3 = 0) \right] \begin{Bmatrix} \mathbf{Lj}_1^{B1}(r) \\ \mathbf{Lj}_2^{B1}(r) \\ \mathbf{Lj}_3^{B1}(r) \end{Bmatrix}$$

where,

$$\begin{aligned}\theta_x^{B1}(r) &= \cos[\theta_s^{B1}(r)]\theta_{IP}^{B1}(r) - \sin[\theta_s^{B1}(r)]\theta_{OoP}^{B1}(r), & \theta_y^{B1}(r) &= \sin[\theta_s^{B1}(r)]\theta_{IP}^{B1}(r) + \cos[\theta_s^{B1}(r)]\theta_{OoP}^{B1}(r), \text{ and} \\ \theta_{IP}^{B1}(r) &= -\left[\frac{d\psi_1^{B1}(r)}{dr}q_{B1F1} + \frac{d\psi_2^{B1}(r)}{dr}q_{B1F2} + \frac{d\psi_3^{B1}(r)}{dr}q_{B1E1}\right], & \theta_{OoP}^{B1}(r) &= \left[\frac{d\phi_1^{B1}(r)}{dr}q_{B1F1} + \frac{d\phi_2^{B1}(r)}{dr}q_{B1F2} + \frac{d\phi_3^{B1}(r)}{dr}q_{B1E1}\right]\end{aligned}$$

The equation for $\mathbf{n}^{B2}(r)$ is similar.

Blade Element-Fixed Coordinate System Used for Calculating and Returning Aerodynamic Loads

This coordinate system is coincident with $\mathbf{i}^{B1}$ when the blade is undeflected.

$$\begin{Bmatrix} \mathbf{m}_1^{B1}(r) \\ \mathbf{m}_2^{B1}(r) \\ \mathbf{m}_3^{B1}(r) \end{Bmatrix} = \begin{bmatrix} \cos[BlPitch(1) + \theta_s^{B1}(r)] & \sin[BlPitch(1) + \theta_s^{B1}(r)] & 0 \\ -\sin[BlPitch(1) + \theta_s^{B1}(r)] & \cos[BlPitch(1) + \theta_s^{B1}(r)] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{n}_1^{B1}(r) \\ \mathbf{n}_2^{B1}(r) \\ \mathbf{n}_3^{B1}(r) \end{Bmatrix}$$

The equation for $\mathbf{m}^{B2}(r)$ is similar.

Blade Element-Fixed Coordinate System Aligned with Local Aerodynamic Axes (i.e., chordline) / Trailing Edge Coordinate System

$$\begin{Bmatrix} \mathbf{te}_1^{B1}(r) \\ \mathbf{te}_2^{B1}(r) \\ \mathbf{te}_3^{B1}(r) \end{Bmatrix} = \begin{bmatrix} \cos[BlPitch(1) + \theta_A^{B1}(r)] & -\sin[BlPitch(1) + \theta_A^{B1}(r)] & 0 \\ \sin[BlPitch(1) + \theta_A^{B1}(r)] & \cos[BlPitch(1) + \theta_A^{B1}(r)] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{m}_1^{B1}(r) \\ \mathbf{m}_2^{B1}(r) \\ \mathbf{m}_3^{B1}(r) \end{Bmatrix}$$

The equation for $\mathbf{te}^{B2}(r)$ is similar.

Tail-Furl Coordinate System

$$\begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{Bmatrix} = \begin{bmatrix} [1 - \cos^2(TFrlSkew)\cos^2(TFrlTilt)]\cos(q_{TFrl}) & \cos(TFrlSkew)\cos(TFrlTilt)\sin(TFrlTilt)[1 - \cos(q_{TFrl})] & \cos(TFrlSkew)\sin(TFrlSkew)\cos^2(TFrlTilt)[\cos(q_{TFrl}) - 1] \\ +\cos^2(TFrlSkew)\cos^2(TFrlTilt) & -\sin(TFrlSkew)\cos(TFrlTilt)\sin(q_{TFrl}) & -\sin(TFrlTilt)\sin(q_{TFrl}) \\ \cos(TFrlSkew)\cos(TFrlTilt)\sin(TFrlTilt)[1 - \cos(q_{TFrl})] & \cos^2(TFrlTilt)\cos(q_{TFrl}) + \sin^2(TFrlTilt) & \sin(TFrlSkew)\cos(TFrlTilt)\sin(TFrlTilt)[\cos(q_{TFrl}) - 1] \\ +\sin(TFrlSkew)\cos(TFrlTilt)\sin(q_{TFrl}) & & +\cos(TFrlSkew)\cos(TFrlTilt)\sin(q_{TFrl}) \\ \cos(TFrlSkew)\sin(TFrlSkew)\cos^2(TFrlTilt)[\cos(q_{TFrl}) - 1] & \sin(TFrlSkew)\cos(TFrlTilt)\sin(TFrlTilt)[\cos(q_{TFrl}) - 1] & [1 - \sin^2(TFrlSkew)\cos^2(TFrlTilt)]\cos(q_{TFrl}) \\ +\sin(TFrlTilt)\sin(q_{TFrl}) & -\cos(TFrlSkew)\cos(TFrlTilt)\sin(q_{TFrl}) & +\sin^2(TFrlSkew)\cos^2(TFrlTilt) \end{bmatrix} \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{Bmatrix}$$

Tail Fin Coordinate System

$$\begin{Bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{Bmatrix} = \begin{bmatrix} \cos(TFinSkew)\cos(TFinTilt) & \sin(TFinTilt) & -\sin(TFinSkew)\cos(TFinTilt) \\ \sin(TFinSkew)\sin(TFinBank) & \cos(TFinTilt)\cos(TFinBank) & \cos(TFinSkew)\sin(TFinBank) \\ -\cos(TFinSkew)\sin(TFinTilt)\cos(TFinBank) & & +\sin(TFinSkew)\sin(TFinTilt)\cos(TFinBank) \\ \sin(TFinSkew)\cos(TFinBank) & -\cos(TFinTilt)\sin(TFinBank) & \cos(TFinSkew)\cos(TFinBank) \\ +\cos(TFinSkew)\sin(TFinTilt)\sin(TFinBank) & & -\sin(TFinSkew)\sin(TFinTilt)\sin(TFinBank) \end{bmatrix} \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{Bmatrix}$$